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ERROR ANALYSIS OF THE TAU METHOD: DEPENDENCE OF THE ERROR ON THE DEGREE AND ON THE LENGTH OF THE INTERVAL OF APPROXIMATION

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Abstract—In this paper, we discuss the dependence of Tau Method approximations on (i) the degree of approximant n and (ii) the length of the interval of approximation h. We shall show that the Tau Method parameters τ_i (i) decay exponentially in terms of n and (ii) for a fixed n they decay in terms of h as $(h/2)^n$.

1. INTRODUCTION

Let D be a differential operator of order ν with polynomial coefficients and let $y(x)$ be the exact solution of the differential equation

$$Dy(x) = f(x); \qquad x \in [-1,1] \tag{1}$$

$$B_j(y) = \gamma_j; \qquad \gamma_j \in \mathbb{R}, \tag{2}$$

where $\mathbb{R}$ indicates the set of real numbers, $\{B_j; j = 1,2,\dots,\nu\}$ are ν linear functionals acting on the elements of $C^{\nu}[-1,1]$, the space of ν-times continuously differentiable functions defined on $[-1,1]$, and $f(x)$ is a polynomial of degree $d_f \in \mathbb{N} := \{0,1,2,\dots\}$. Let us assume that $y_n(x)$, $n \in \mathbb{N}$, is a Tau Method approximation of (1)–(2) (see [1]); that is, $y_n(x)$ satisfies *exactly* a perturbed problem of the form

$$\begin{aligned} Dy_n(x) &= f(x) + H_n(x); \qquad x \in [-1,1],\ n \in \mathbb{N} \\ B_j(y_n) &= \gamma_j; \qquad j = 1,2,\dots,\nu, \end{aligned} \tag{3}$$

where $H_n(x)$ is a polynomial perturbation term which reduces the *exact* solution of (3) to a polynomial $y_n(x)$. The order of $H_n(x)$ depends on a finite number of free parameters, say $\{\tau_i \in \mathbb{R}; i = 0,1,\dots,r \in \mathbb{N}\}$ which are required to satisfy conditions (2), and to match the residual component of $Dy_n(x)$ with that of the right hand side of (3) (see [1]). Let us assume that $H_n(x) := \left(\sum_{i=0}^{r-1} \tau_i x^i\right) V_n(x)$, where $V_n(x)$ is a Chebyshev or Legendre polynomial of degree n. Then, the error function $e_n(x) := y(x) - y_n(x)$ satisfies

$$\|e_n\|_{[-1,1]} := \max_{-1\le x\le 1} |e_n(x)| \le 2\kappa \sum_{j=0}^{r-1} |\tau_{j,n}|, \tag{4}$$

where κ is an upper bound of the Green's function, say $G(x,t)$, associated with the given differential operator D. From the properties of the Green's function (see [2]), $G(x,t)$ is always continuous on the compact $[-1,1] \times [-1,1]$, and consequently κ exits, is finite and independent of n. Then taking limits in (4) we find that

$$\lim_{n\longrightarrow\infty} \frac{\|e_n\|_{[-1,1]}}{\sum_{j=0}^{r-1} |\tau_{j,n}|} \le 2\kappa.$$

Thus, $\|e_n\|_{[-1,1]}$ and $\sum_{j=0}^{r-1} |\tau_{j,n}|$ have the same rate of convergence provided that the coefficients of the given differential operator are sufficiently smooth. Consequently, the size of the τ_i's plays a crucial role in estimating the error of Tau Method approximations.

In this paper, we shall investigate the speed of convergence of the approximation error by concentrating on the behavior of these parameters. In Section 2, we discuss the dependence of the Tau parameters on the order of approximation n; our main result, given in Theorem 3, states that *these parameters decay exponentially in terms of* n. This question was first investigated by Namasivayam and Ortiz in [3, 4] who considered systems of differential equations with constant coefficients. Our results apply to a particular but important class of second order differential equations which includes those with variable coefficients. In Section 3, we fix n and alternatively, study the dependence of Tau parameters on h, the measure of the interval on which the approximation is sought. The main result of this section is Theorem 2, which states that *the Tau parameters decay as* $2^{-n}h^n$. In Section 4, we give numerical applications which illustrate the accuracy of our estimates.

2. DEPENDENCE OF TAU PARAMETERS ON THE ORDER OF APPROXIMATION

Let us consider the second order ordinary differential equation

$$Dy(x) := -y''(x) + ax^\alpha y(x) = f(x); \qquad x \in [-1,1], \tag{5}$$

$$y(-1) = \gamma_1, \quad y(1) = \gamma_2, \tag{6}$$

where $\{a, \gamma_1, \gamma_2\} \subset \mathbb{R}$, $\alpha \in \mathbb{N}$ and $f(x) := \sum_{i=0}^{d_f} f_i x^i$ is a polynomial of degree d_f.

The sequence of canonical polynomials associated with D is generated recursively by

$$Q_n(x) = \frac{1}{a}\, x^{n-\alpha} + \frac{1}{a}\, \frac{(n-\alpha)!}{(n-\alpha-2)!}\, Q_{n-\nu-2}(x), \quad n \geq \alpha. \tag{7}$$

The first α canonical polynomial $Q_0(x), Q_1(x), \dots, Q_{\alpha-1}(x)$ cannot be defined by (7); hence, the set of powers of x inaccessible through applying D to polynomials is given by $S = \{0, 1, 2, \dots, \alpha - 1\}$.

For all $k \in S$ and $n \in \mathbb{N}$ let $\rho_k^{(n)}$ denote the coefficient of $Q_k(x)$ in the expression of $Q_n(x)$. From (7), $\{\rho_k^{(n)};\ n \in \mathbb{N}, k \in S\}$ are generated by

$$\rho_k^{(n)} = \delta_k^n; \qquad k \in S,\ n \in S \tag{8}$$

and

$$\rho_k^{(n)} = \frac{1}{a}\, \frac{(n-\alpha)!}{(n-\alpha-2)!}\, \rho_k^{(n-\alpha-2)}; \qquad k \in S,\ n \geq \alpha. \tag{9}$$

With (5)–(6) we associate the Tau problem

$$\begin{aligned} Dy_N(x) &= f(x) + H_N(x); \qquad x \in [-1,1],\ N \in \mathbb{N} \\ y_N(-1) &= \gamma_1, \quad y_N(1) = \gamma_2, \end{aligned} \tag{10}$$

where $H_N(x)$ is a polynomial perturbation term which forces the exact solution of (10) to become a polynomial. Since $y(x)$ is subjected to two boundary conditions (6); card$(S) = \alpha$ and $\mathrm{Ker}_D = \emptyset$, $H_N(x)$ must involve $\alpha + 2$ free parameters $\{\tau_{j,N},\ j = 0, 1, \dots, \alpha + 1\}$. Let us choose

$$H_N(x) := \left(\sum_{j=0}^{\alpha+1} \tau_{j,N} x^j \right) V_N(x) \quad \text{where } V_N(x) := \sum_{n=0}^{N} c_n^N x^n,\ \ c_N^N \neq 0. \tag{11}$$

From Theorem 3 of Ortiz [1] we have

$$y_N(x) = \sum_{\substack{i=0 \\ i \notin S}}^{d_f} f_i\, Q_i(x) + \sum_{j=0}^{\alpha+1} \tau_{j,N} \left[\sum_{\substack{n=0 \\ n+j \notin S}}^{N} c_n^N Q_{n+j}(x) \right].$$

Taking into account the boundary conditions and the S-Matching conditions of the Tau Method (see [5]), we can construct the algebraic system which gives the values of $\{\tau_{i,N}; i = 0, 1, \ldots, \alpha+1\}$:

$$\mathbf{M}\underline{\underline{\tau_N}} = \underline{\underline{f_\gamma}} \tag{12}$$

where

$$\mathbf{M} = \begin{bmatrix} \sum_{n=0}^{N} c_n^N \rho_0^{(n)} & \sum_{n=0}^{N} c_n^N \rho_0^{(n+1)} & \cdots & \sum_{n=0}^{N} c_n^N \rho_0^{(n+\alpha+1)} \\ \sum_{n=0}^{N} c_n^N \rho_1^{(n)} & \sum_{n=0}^{N} c_n^N \rho_1^{(n+1)} & \cdots & \sum_{n=0}^{N} c_n^N \rho_1^{(n+\alpha+1)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{n=0}^{N} c_n^N \rho_{\alpha-1}^{(n)} & \sum_{n=0}^{N} c_n^N \rho_{\alpha-1}^{(n+1)} & \cdots & \sum_{n=0}^{N} c_n^N \rho_{\alpha-1}^{(n+\alpha+1)} \\ \sum_{n=0}^{N} c_n^N Q_n(-1) & \sum_{n=0}^{N} c_n^N Q_{n+1}(-1) & \cdots & \sum_{n=0}^{N} c_n^N Q_{n+\alpha+1}(-1) \\ \sum_{n=0}^{N} c_n^N Q_n(1) & \sum_{n=0}^{N} c_n^N Q_{n+1}(1) & \cdots & \sum_{n=0}^{N} c_n^N Q_{n+\alpha+1}(1) \end{bmatrix},$$

$$\underline{\underline{\tau_N}} := \left[\tau_{0,N}, \tau_{1,N}, \ldots, \tau_{\alpha+1,N}\right]^{\mathsf{T}},$$

and

$$\underline{\underline{f_\gamma}} = \left[-\sum_{i=0}^{d_f} f_i \rho_0^{(i)}, \quad -\sum_{i=0}^{d_f} f_i \rho_1^{(i)}, \quad \cdots \quad \cdots \quad -\sum_{i=0}^{d_f} f_i \rho_{\alpha+1}^{(i)}, \quad \gamma_1, \quad \gamma_2 \right]^{\mathsf{T}}.$$

Let $\mathbf{M}^{(\ell)}$ indicate the matrix obtained from $\mathbf{M}$ replacing its ℓ^{th} column by $\underline{\underline{f_\gamma}}$. In this notation, the solution of (12) is

$$\tau_{\ell,N} = \frac{\det \mathbf{M}^{(\ell)}}{\det \mathbf{M}} \quad \forall \ell \in \{0, 1, \ldots, \alpha+1\}. \tag{13}$$

We shall now find the orders of $\{\tau_{\ell,N}; \ell = 0, 1, \ldots, \alpha+1\}$ in terms of N. The strategy we shall follow is to determine first the orders of $\det \mathbf{M}^{(\ell)}$ and $\det \mathbf{M}$. We require the following two lemmas.

Lemma 1. *For all $n \geq \alpha$ there exist two uniquely determined integers $K_n \in \mathbb{N}$ and $\vartheta_n \in \{0, 1, 2, \ldots, \alpha+1\}$ such that*

$$n = (\alpha+2)K_n + \alpha + \vartheta_n. \tag{14}$$

Proof. We use an inductive argument: when $n = \alpha$, we take $K_n = \vartheta_n = 0$; hence, (14) holds for $n = \alpha$. Assuming it is true for some $n \geq \alpha$, i.e., $n = (\alpha+2)K_n + \alpha + \vartheta_n$, let us prove it holds for $n+1$. We have $n + 1 = (\alpha+2)K_n + \alpha + \vartheta_n + 1$. Two cases to be discussed follow.

First, if $\vartheta_n < \alpha+1$ then $\vartheta_n + 1 \leq \alpha+1$ and, therefore, we take $K_{n+1} := K_n$ and $\vartheta_{n+1} := \vartheta_n + 1$.

On the other hand, if $\vartheta_n = \alpha+1$ then $n+1 = (\alpha+2)(K_n+1) + \alpha$; hence $K_{n+1} := K_n + 1$ and $\vartheta_{n+1} := 0$.

Finally, to show the uniqueness of representation (1): let us suppose that $n = (\alpha+2)K^* + \alpha + \vartheta^*$ where $K^* \neq K_N$ and $\vartheta^* \neq \vartheta_N$, then

$$(K_n - K^*)(\alpha+2) + (\vartheta_n - \vartheta^*) = 0. \tag{15}$$

Assuming, without loss of generality, that $\vartheta_n - \vartheta^* > 0$ and $K_n - K^* < 0$ we can write

$$(K_n - K_n^*)(\alpha+2) + (\vartheta_n - \vartheta_n^*) < (K_n - K_n^*)(\alpha+2) + \alpha + 1 < -(\alpha+2) + \alpha + 1 < -1,$$

which contradicts (15). ∎

For the next lemma, we need Landau's order notation: if $f(x)$ and $g(x)$ are two functions of $x \in \mathbb{R}$ and if $x_0 \in \mathbb{R}$, we shall say that $f(x) = O(g(x))$ near x_0 if there exists a constant $\kappa \in \mathbb{R}$ such that $\lim_{x \to x_0} |f(x)/g(x)| \leq \kappa$. Throughout this section, orders will be taken near $+\infty$.

The following lemma, the proof of which is given in Appendix A1, gives *explicit* expressions for the canonical polynomials $\{Q_n(x); n \in \mathrm{N} - S\}$ in terms of powers of x.

LEMMA 2. *Under the above assumptions we have:*

1. *For all $n \geq \alpha$, the canonical polynomial associated with the differential operator (5) are given by*

$$Q_n(x) = \frac{1}{a} x^{n-\alpha} + \sum_{k=1}^{\frac{n-\alpha-\vartheta_n}{\alpha+2}} \frac{1}{a^{k+1}} \left\{ \prod_{j=1}^{k} \frac{[n-(j-1)(\alpha+2)-\alpha]!}{[n-j(\alpha+2)]!} \right\} x^{n-k(\alpha+2)-\alpha}.$$

Equivalently,

$$Q_n(x) = \frac{1}{a} x^{n-\alpha} + \sum_{k=1}^{K_n} \frac{1}{a^{k+1}} \left\{ \prod_{j=1}^{k} \frac{[(K_n-j+1)(\alpha+2)+\vartheta_n]!}{[(K_n-j)(\alpha+2)+\vartheta_n+\alpha]!} \right\} x^{(K_n-k)(\alpha+2)+\vartheta_n}. \tag{16}$$

2. *For all $n \geq \alpha$*

$$Q_n(\pm 1) = O\left(\frac{1}{a^{K_n+1}} \prod_{j=1}^{K_n} \frac{[(K_n-j+1)(\alpha+2)+\vartheta_n]!}{[(K_n-j)(\alpha+2)+\vartheta_n+\alpha]!} \right). \tag{17}$$

3. *For all $n \geq \alpha$ and all $i \in S$*

$$\rho_i^{(n)} = O\left(\frac{1}{a^{K_n+1}} \prod_{j=1}^{K_n} \frac{[(K_n-j+1)(\alpha+2)+\vartheta_n]!}{[(K_n-j)(\alpha+2)+\vartheta_n+\alpha]!} \right). \tag{18}$$

We can now state and prove the main result of this section.

THEOREM 1. *Under the above assumptions we have the following equations.*

1. *For all $i \in \{0, 1, \dots, \alpha+1\}$*

$$\tau_{i,N} = O\left(\frac{a^{K_{(N+i)}+1}}{c_N^N} \prod_{j=1}^{K_{(N+i)}} \frac{[(K_{(N+i)}-j)(\alpha+2)+\vartheta_{(N+i)}+\alpha]!}{[(K_{(N+i)}-j+1)(\alpha+2)+\vartheta_{(N+i)}]!} \right). \tag{19}$$

2. *Then for all $i \in \{0, 1, \dots, \alpha+1\}$ and sufficiently large $N \in \mathbb{N}$,*

$$\frac{\tau_{i+1,N}}{\tau_{i,N}} < O(1). \tag{20}$$

3. *For sufficiently large $N := (\alpha+2)K + \alpha$ the function*

$$\Upsilon(N) := \frac{a^{K+1}}{c_N^N} \prod_{j=1}^{K} \frac{[(K-j)(\alpha+2)+\alpha]!}{[(K-j+1)(\alpha+2)]!} \tag{21}$$

satisfies

$$\Upsilon(N) \leq \frac{a^{K+1}}{c_N^N (\alpha+2)^{2K-1}[(K-1)!]^2} \tag{22}$$

and

$$\lim_{N \to +\infty} \Upsilon(N) = 0.$$

Before proceeding, we shall make three observations on the statement of Theorem 1.

(i) $\Upsilon(N)$ is the right hand side of (19) when $i = 0$; therefore, its behavior will enable us to monitor the variations of $\tau_{0,N}$ for sufficiently large N,

(ii) (20) means that, for sufficiently large $N \in \mathbb{N}$, $\{\tau_{i,N}, i = 0,1,\ldots,\alpha+1\}$ form a *strictly decreasing sequence* whose $\tau_{0,N}$ is the largest term; that is, $\tau_{0,N}$ is the *dominant Tau parameter*, and

(iii) from (22) we deduce that $\tau_{0,N}$, and hence all the $\tau_{i,N}$'s, converge to zero more rapidly than $a^{K+1}\left(c_N^N(\alpha+2)^{2K-1}[(K-1)!]^2\right)^{-1}$.

PROOF.

1. When N is sufficiently large the following identities hold.

$$\sum_{n=0}^{N} c_n^N Q_{n+i}(\pm 1) = O\left(c_N^N Q_{N+i}(\pm 1)\right) \tag{23}$$

and

$$\sum_{n=0}^{N} c_n^N \rho_j^{(n+i)} = O\left(c_N^N \rho_j^{(N+i)}\right), \tag{24}$$

where $i \in \{0,1,2,\ldots,\alpha+1\}$ and $j \in S$. Substituting (17) in (23) and (18) in (24), we find that

$$\det \mathbf{M} = O\left[\prod_{i=0}^{\alpha+1}\left(\frac{1}{a^{K_{(N+i)}+1}}\prod_{j=1}^{K_{N+i}}\frac{[(K_{N+i}-j+1)(\alpha+2)+\vartheta_{N+i}]!}{[(K_{N+i}-j)(\alpha+2)+\vartheta_{N+i}+\alpha]!}\right)\right]. \tag{25}$$

Similarly

$$\det \mathbf{M}^{\ell} = O\left[\prod_{\substack{i=0\\ i\neq \ell}}^{\alpha+1}\left(\frac{1}{a^{K_{(N+i)}+1}}\prod_{j=1}^{K_{N+i}}\frac{[(K_{N+i}-j+1)(\alpha+2)+\vartheta_{N+i}]!}{[(K_{N+i}-j)(\alpha+2)+\vartheta_{N+i}+\alpha]!}\right)\right]. \tag{26}$$

Combining (25), (26) and (13) we get (19).

2. By hypothesis $N := (\alpha+2)K+\alpha$; that is, $K_N = K$ and $\vartheta_N = 0$. Therefore, for all $i \in \{0,1,\ldots,\alpha+1\}$

$$N + i = (\alpha+2)K+\alpha+i \text{ and hence } K_{(N+i)} = K \text{ and } \vartheta_{(N+i)} = i.$$

From (19),

$$\begin{aligned}
\frac{\tau_{i+1,N}}{\tau_{i,N}} &= O\left(\prod_{j=1}^{K}\frac{[(K-j)(\alpha+2)+\alpha+1+i][(K-j)(\alpha+2)+\alpha+i]}{[(K-j)(\alpha+2)+\alpha+2+i][(K-j)(\alpha+2)+\alpha+1+i]}\right)\\
&= O\left(\prod_{j=1}^{K}\frac{[(K-j)(\alpha+2)+\alpha+i]}{[(K-j)(\alpha+2)+\alpha+2+i]}\right)\\
&= O\left(\frac{\alpha+i}{\alpha+i+2}\prod_{j=1}^{K-1}\frac{[(K-j)(\alpha+2)+\alpha+i]}{[(K-j)(\alpha+2)+\alpha+2+i]}\right) < O(1)
\end{aligned}$$

and hence (20) holds.

3. Let $i = 0$ in (19); rearranging terms

$$\begin{aligned}
\Upsilon(N) &= \frac{a^{K+1}}{c_N^N} \prod_{j=1}^{K} \frac{1}{[(K-j)(\alpha+2)+\alpha+2][(K-j)(\alpha+2)+\alpha+1]} \\
&= \frac{a^{K+1}}{c_N^N(\alpha+2)^K(\alpha+1)} \prod_{j=1}^{K-1} \frac{1}{(K-j)^2(\alpha+2)+(K-j)((\alpha+1)+(\alpha+2))+(\alpha+1)} \\
&\le \frac{a^{K+1}}{c_N^N(\alpha+2)^K(\alpha+1)} \prod_{j=1}^{K-1} \frac{1}{(K-j)^2(\alpha+2)} \\
&= \frac{a^{K+1}}{c_N^N(\alpha+2)^K(\alpha+1)} \frac{1}{[(K-1)!]^2(\alpha+2)^{K-1}} = \frac{a^{K+1}}{c_N^N(\alpha+2)^{2K-1}[(K-1)!]^2},
\end{aligned}$$

as required.

It remains to verify that the left hand side of (22) converges to zero when N tends to $+\infty$. This can be justified showing that

$$\lim_{K\to\infty} \frac{a^{K+1}}{(K-1)!} = 0 \quad \text{or} \quad \lim_{K\to\infty} \ln \frac{a^{K+1}}{(K-1)!} = -\infty.$$

For this we need Stirling's formula: $n! \approx e^{-n} n^{n+1/2}\sqrt{2\pi}$ for large $n \in \mathbb{N}$. Applying it to $(K-1)!$ we find that

$$\begin{aligned}
\ln \frac{a^{K+1}}{(K-1)!} &\approx (K+1)\ln a - \ln\left[e^{-(K-1)}(K-1)^{K-\frac{1}{2}}\sqrt{2\pi}\right] \\
&= (K+1)\ln a + K - 1 - \left(K-\frac{1}{2}\right)\ln(K-1) - \ln\sqrt{2\pi} \\
&= -\left(K-\frac{1}{2}\right)\ln(K-1) + (1+\ln a)K - 1 + \ln a - \ln\sqrt{2\pi},
\end{aligned}$$

which tends to $-\infty$ when K tends to $+\infty$ as $\lim_{K\to\infty} \frac{(K-1/2)\log(K-1)}{K} = +\infty$. ∎

COROLLARY 1. *Suppose that $\alpha = 0$. Then for any $a \in \mathbb{R}$ we have*

$$\tau_{0,N} = O\left(\frac{a^{N/2+1}}{N!\,c_N^N}\right) \quad \text{and} \quad \tau_{1,N} = O\left(\frac{a^{N/2+1}}{(N+1)!\,c_N^N}\right). \tag{27}$$

PROOF. Let $N = 2K$ i.e., $K_N = K$ and $\vartheta_N = 0$. Assuming that $\alpha = 0$, (19) is reduced to

$$\begin{aligned}
\tau_{i,N} &= O\left(\frac{a^{N/2+1}}{C_N^N} \prod_{j=1}^{N/2} \frac{[2(N/2-j)+i]!}{[2(N/2-j)+i+2]!}\right) \\
&= O\left(\frac{[2(N/2-j)+i]!}{[2(N/2-j)+i+2]!}\right) = O\left(\frac{a^{N/2+1}}{c_N^N}\frac{1}{(N+i)!}\right); \quad i = 0, 1,
\end{aligned}$$

which, in turn, gives (27). ∎

We wish to point out at this stage that, if we consider instead of equation (5), equations of the form

$$y''(x) + (ax^\alpha + \mu(x))\,y(x) = f(x),$$

where $\mu(x)$ is a polynomial, the estimates of Theorem 1 remain valid, provided that $\mu(x)$ has a degree relatively small compared with that of α. In Section 4, we consider a concrete numerical example where such condition is satisfied.

3. DEPENDENCE OF TAU PARAMETERS ON THE LENGTH OF THE INTERVAL OF APPROXIMATION

Let us consider the IVP defined by the following second order differential equations with variable coefficients

$$Dy(x) := y''(x) + B(x)y'(x) + C(x)y(x) = f(x); \qquad x \in [x_0, x_0 + h] \tag{28}$$

$$y(x_0) = \gamma_0, \;\; y'(x_0) = \gamma_2, \tag{29}$$

where $B(x) := \sum_{i=0}^{\alpha} b_i x^i$, $C(x) := \sum_{i=0}^{\beta} c_i x^i$ and $f(x) := \sum_{i=0}^{d_f} f_i x^i$.

We shall now investigate the dependence of Tau parameters on h while the order of the perturbation term remains unchanged.

For simplicity we will assume that $\alpha \leq \beta$. We can also assume, without loss of generality, that $c_\beta = 1$. The canonical polynomials associated with (28) are generated recursively through

$$Q_{n+\beta} = x^n - n(n-1)Q_{n-2} - \sum_{j=0}^{\alpha} nb_j\, Q_{n+i-1} - \sum_{j=0}^{\beta-1} c_j\, Q_{n+j}; \qquad n \in \mathbb{N}. \tag{30}$$

Following the arguments of Section 2, we find that $S = \{0, 1, \ldots, \beta-1\}$ and Ker D $= \emptyset$. From (30), $\{\rho_k^m; m \in \mathbb{N}, k \in \mathrm{S}\}$ is defined recursively by

$$\rho_k^{(n+\beta)} = -n(n-1)\rho_k^{(n-2)} - \sum_{j=0}^{\alpha} nb_j \rho_k^{(n+i-1)} - \sum_{j=0}^{\beta-1} c_j \rho_k^{(n+j)}; \qquad n \in \mathbf{N}.$$

Let $V_{N,h}(x) := \sum_{i=0}^{N} c_{i,h}^N x^i$ be a transformation of (11) to the interval $[x_0, x_0 + h]$. Let us define $\mathbf{h} := h/2$. Then

$$V_{N,h}(x) = V_N\left(\frac{x - s_0}{\mathbf{h}}\right) \text{ where } s_0 = x_0 + \mathbf{h}.$$

Let us assume that a perturbation of form

$$H_N(x) = \left(\sum_{j=0}^{r-1} \tau_{j,h} x^j\right) V_{N,h}(x); \;\; x \in [x_0, x_0 + h]; \qquad r = \beta + 2 \tag{31}$$

is used to solve (28)–(29) with the Tau Method. We will prove the following result.

Theorem 2. *For all $j = 0, 1, 2, \ldots, r-1$ the Tau parameters $\tau_{j,h}$ of (31) satisfy*

$$\tau_{j,h} = O(\mathbf{h}^N).$$

Equivalently, there exists $\theta_j(\mathbf{h}) = O(1)$ such that

$$\tau_{j,h} = \theta_j \mathbf{h}^N + O(\mathbf{h}^{N+1}). \tag{32}$$

We need the following lemma, the proof of which is given in Appendix A2.

Lemma 3. *The following statements are true.*

1. Given any two sequences $\{a_i\}$ and $\{b_{ij}\}$ we have

$$\sum_{j=0}^{n} a_j \sum_{i=0}^{j} b_{ij}\, x^i = \sum_{j=0}^{n} \left[\sum_{i=j}^{n} a_i\, b_{ji}\right] x^j \text{ and } \sum_{i=j}^{n} a_i \sum_{k=0}^{i-j} b_{ik}\, x^k = \sum_{i=0}^{n-j} \left[\sum_{k=i+j}^{n} a_k\, b_{ki}\right] x^i.$$

2. *For all* $j = 0, 1, \ldots, N$

$$c_{j,h}^N = \sum_{i=0}^{N-j}\left[\sum_{k=i+j}^{N} a_k\, b_{ki}\right]\mathbf{h}^i, \tag{33}$$

where

$$a_k = \frac{c_k^N}{\mathbf{h}^k}\binom{k}{j}(-1)^{(k-j)} \text{ and } b_{ki} = \binom{k-j}{i}x_0^{k-j-i}. \tag{34}$$

3. *For all* $j = 0, 1 \ldots, N$ *we have*

$$c_{j,h}^N = \frac{K_j}{\mathbf{h}^N} + O\left(\frac{1}{\mathbf{h}^{N-1}}\right) \text{ if } x_0 \neq 0,$$

where

$$K_j = (-1)^{N-j}\binom{N}{j}c_N^N(-x_0)^{N-j}. \tag{35}$$

We now proceed to the proof of Theorem 2. From [1] it follows that the $\tau_{j,h}$'s satisfy

$$\sum_{j=0}^{r-1}\tau_{j,h}\left[\sum_{i=0}^{N} c_{i,h}^N\,\rho_k^{(i+j)}\right] = -\sum_{j=0}^{d_f} f_j\,\rho_k^{(j)}, \qquad \forall k \in S \tag{36}$$

$$\sum_{j=0}^{r-1}\tau_{j,h}\left[\sum_{i=0}^{N} c_{i,h}^N\,Q_{i+j}(x_0)\right] = \gamma_0 - \sum_{j=0}^{d_f} f_j\,Q_j(x_0) \tag{37}$$

$$\sum_{j=0}^{r-1}\tau_{j,h}\left[\sum_{i=0}^{N} c_{i,h}^N\,Q'_{i+j}(x_0)\right] = \gamma_1 - \sum_{j=0}^{d_f} f_j\,Q'_j(x_0). \tag{38}$$

Combining (36), (37) and (38) we find that

$$\mathbf{M}\underline{\underline{\tau_h}} = \underline{\underline{f_\gamma}}$$

where

$$\mathbf{M} := \begin{pmatrix} \sum_{i=0}^{N} c_{i,h}^N\,\rho_0^{(i)} & \sum_{i=0}^{N} c_{i,h}^N\,\rho_0^{(i+1)} & \cdots & \sum_{i=0}^{N} c_{i,h}^N\,\rho_0^{(i+r-1)} \\ \sum_{i=0}^{N} c_{i,h}^N\,\rho_1^{(i)} & \sum_{i=0}^{N} c_{i,h}^N\,\rho_1^{(i+1)} & \cdots & \sum_{i=0}^{N} c_{i,h}^N\,\rho_1^{(i+r-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=0}^{N} c_{i,h}^N\,\rho_{s-1}^{(i)} & \sum_{i=0}^{N} c_{i,h}^N\,\rho_{s-1}^{(i+1)} & \cdots & \sum_{i=0}^{N} c_{i,h}^N\,\rho_{s-1}^{(i+r-1)} \\ \sum_{i=0}^{N} c_{i,h}^N\,Q_i(x_0) & \sum_{i=0}^{N} c_{i,h}^N\,Q_{i+1}(x_0) & \cdots & \sum_{i=0}^{N} c_{i,h}^N\,Q_{i+r-1}(x_0) \\ \sum_{i=0}^{N} c_{i,h}^N\,Q'_i(x_0) & \sum_{i=0}^{N} c_{i,h}^N\,Q'_{i+1}(x_0) & \cdots & \sum_{i=0}^{N} c_{i,h}^N\,Q'_{i+r-1}(x_0) \end{pmatrix}$$

$$\underline{\underline{\tau_h}} := (\tau_0, \tau_1, \tau_2, \ldots, \tau_{r-1})^{\mathsf{T}}$$

and

$$\underline{\underline{f_\gamma}} := \left(-\sum_{j=0}^{d_f} f_j\,\rho_0^{(j)}, \ldots, -\sum_{j=0}^{d_f} f_j\,\rho_{s-1}^{(j)}, \gamma_0 - \sum_{j=0}^{d_f} f_j\,Q_j(x_0), \gamma_1 - \sum_{j=0}^{d_f} f_j\,Q'_j(x_0)\right)^{\mathsf{T}}.$$

Let $\mathbf{M}_{mn}$ designate any item with index (m,n) in $\mathbf{M}$. When $m \le s-1$ we have

$$\begin{aligned}\mathbf{M}_{mn} &:= \sum_{i=0}^{N} c_{i,h}^{N}\,\rho_m^{(i+n)} = \sum_{i=0}^{N}\left[\frac{K_i}{\mathbf{h}^N} + O\left(\frac{1}{\mathbf{h}^{N-1}}\right)\right]\rho_m^{(i+n)} \\ &= \frac{1}{\mathbf{h}^N}\left[\sum_{i=0}^{N} K_i\,\rho_m^{(i+n)}\right] + O\left(\frac{1}{\mathbf{h}^{N-1}}\right) \equiv O\left(\frac{1}{\mathbf{h}^N}\right).\end{aligned}$$

Consequently,

$$\det \mathbf{M} := \sum_{\sigma\in\Gamma_{r-1}} \mathbf{M}_{0\sigma_0}\mathbf{M}_{1\sigma_1}\dots\mathbf{M}_{r-1\sigma_{r-1}} = \left[O\left(\frac{1}{\mathbf{h}^N}\right)\right]^r = O\left(\frac{1}{\mathbf{h}^{rN}}\right),$$

where Γ_m is the permutation group of $\{0,1,2,\dots,m\}$. Similarly for any submatrix $\mathbf{M}^{(j)}$ obtained from $\mathbf{M}$ by replacing its j^{th} column by vector $\underline{\underline{f_\gamma}}$ we have

$$\det \mathbf{M}^{(j)} = O\left(\frac{1}{\mathbf{h}^{(r-1)N}}\right).$$

Hence,

$$\tau_{j,h} := \frac{\det \mathbf{M}^{(j)}}{\det \mathbf{M}} = O\left(\mathbf{h}^N\right). \qquad \blacksquare$$

4. NUMERICAL EXAMPLES

EXAMPLE 1. Let us consider the following boundary value problem defined by the second order ordinary differential equation

$$-y''(x) + (2x + x^4)y(x) = 0; \qquad x \in [-1,1] \tag{39}$$

$$y(-1) = \exp\left(-\frac{1}{3}\right),\ y(1) = \exp\left(\frac{1}{3}\right) \tag{40}$$

of which $y(x) = \exp(x^3/3)$ is the exact solution, and let us associate to (39)–(40) the Tau problem

$$\begin{aligned}&-y_N''(x) + (2x + x^4)y_N(x) = H_N(x); \qquad x \in [-1,1] \\ &y_N(-1) = \exp\left(-\frac{1}{3}\right),\ y_N(1) = \exp\left(\frac{1}{3}\right).\end{aligned}$$

Since (39) has four undefined canonical polynomials and two boundary conditions are imposed, we need six free parameters. Let

$$H_N(x) := \left(\sum_{i=0}^{5} \tau_{i,N}x^i\right) T_N(x).$$

According to Theorem 1

$$\tau_{0,N} = O(\Upsilon(N)) \tag{41}$$

where

$$\Upsilon(N) := \frac{4^{K_N+1}}{c_N^N} \prod_{j=1}^{K_N} \frac{[6(K_N-j)+4]!}{[6(K_N-j)+6]!}. \tag{42}$$

Table 1 displays the following results.

COLUMN 2. The exact values of $\Upsilon(N)$ for $N = 10(6)208$. (Notice that for such choices of N, the latter becomes of the form $6K+4$ where $K_N = 1,2,\dots,34$ i.e., $K_N = (N-4)/6$ and $\vartheta_N = 0$).

COLUMN 3. The exact values of $\tau_{0,N}$ for $N = 10(208)6$.

COLUMN 4. The absolute value of the maximum of $e_N(x) := y(x) - y_N(x)$.

COLUMN 5. The ratio $\Upsilon(N)/\tau_{0,N}$ obtained dividing the values listed in Column 2 by those of Column 3 respectively. We notice that this ratio remains almost constant when N becomes very large, as estimation (19) predicted.

In addition, $\ln \Upsilon(N)$, $\ln |\tau_{0,N}|$ and $\ln \|e_N\|$ are plotted in Figure 1 for discrete values of N. We see that $\Upsilon(N)$, $|\tau_{0,N}|$ and $\|e_N\|$ behave approximately like $\exp(-\gamma N)$ for some $\gamma > 0$.

Table 1. The Tau Method has been applied to (39)–(40) using a perturbation term $H_N(x) = \left(\sum_{i=0}^{5} \tau_{i,N} x^i\right) T_N(x)$: The exact values of $\tau_{0,N}$, $\Upsilon(N)$ and their ratios are given for $N = 10(6)208$. The absolute value of the maximum errors is also shown.

N	$\Upsilon(N)$	$\tau_{0,N}$	$\|e_N\|_{[-1,1]}$	$\frac{\Upsilon(N)}{\tau_{0,N}}$	N	$\Upsilon(N)$	$\tau_{0,N}$	$\|e_N\|_{[-1,1]}$	$\frac{\Upsilon(N)}{\tau_{0,N}}$
10	6.5E-5	-1.4E-4	1.4E-6	-4.687E-1	112	1.7E-93	-9.2E-93	7.4E-97	-1.805E-1
16	7.7E-9	-1.9E-8	7.6E-11	-4.075E-1	118	2.0E-99	-1.1E-98	8.3E-103	-1.753E-1
22	3.9E-13	-1.1E-12	2.2E-15	-3.666E-1	124	2.2E-105	-1.3E-104	8.5E-109	-1.705E-1
28	1.1E-17	-3.3E-17	4.3E-20	-3.364E-1	130	2.2E-111	-1.3E-110	7.8E-115	-1.659E-1
34	2.0E-22	-6.4E-22	5.5E-25	-3.127E-1	136	2.0E-117	-1.2E-116	6.6E-121	-1.615E-1
40	2.5E-27	-8.4E-27	5.3E-30	-2.932E-1	142	1.6E-123	-1.0E-122	5.2E-127	-1.574E-1
46	2.2E-32	-8.1E-32	3.8E-35	-2.769E-1	148	1.2E-129	-8.1E-129	3.7E-133	-1.535E-1
52	1.6E-37	-5.9E-37	2.2E-40	-2.628E-1	154	8.7E-137	-5.8E-135	2.4E-139	-1.498E-1
58	8.5E-43	-3.4E-42	1.0E-45	-2.504E-1	160	5.6E-142	-3.8E-141	1.5E-145	-1.463E-1
64	3.7E-48	-1.6E-47	3.8E-51	-2.395E-1	166	3.4E-148	-2.3E-147	8.6E-152	-1.429E-1
70	1.4E-53	-5.9E-53	1.2E-56	-2.296E-1	172	1.9E-154	-1.3E-153	4.5E-158	-1.397E-1
76	4.2E-59	-1.9E-58	3.3E-62	-2.207E-1	178	9.7E-161	-7.1E-160	2.2E-164	-1.366E-1
82	1.1E-64	-5.1E-64	7.6E-68	-2.126E-1	184	4.7E-167	-3.5E-166	1.0E-170	-1.337E-1
88	2.4E-70	-1.2E-69	1.5E-73	-2.052E-1	190	2.1E-173	-1.6E-172	4.6E-177	-1.309E-1
94	4.7E-74	-2.4E-75	2.7E-79	-1.983E-1	196	9.1E-180	-7.1E-179	1.8E-183	-1.282E-1
100	8.1E-82	-4.2E-81	4.2E-85	-1.920E-1	202	3.7E-186	-2.9E-185	7.1E-190	-1.256E-1
106	1.2E-87	-6.6E-87	5.9E-91	-1.861E-1	208	1.4E-192	-1.1E-191	2.6E-196	-1.230E-1

EXAMPLE 2. Let us consider again Example 1, but now as an initial value problem defined for $0 \le x \le h$:

$$\begin{aligned} &-y''(x) + (2x + x^4)y(x) = 0; \qquad x \in [0, h] \\ &y(0) = 1,\ y'(0) = 0; \end{aligned} \tag{43}$$

we associate with it the Tau problem

$$\begin{aligned} &-y_N''(x) + (2x + x^4)y_N(x) = H_N(x); \qquad x \in [0, h] \\ &y_N(0) = 1,\ y_N'(0) = 0. \end{aligned} \tag{44}$$

Let $H_N(x)$ be defined as before, with $V_{N,h}(x) := T_{N,h}(x)$ and let $N = 6$. Below we list the expression of $\{\tau_{j,h}, j = 0, 1, \dots, 5\}$ in terms of h. All of these parameters are of $O(h^6)$ at least, in agreement with the statement of Theorem 2.

$$\begin{aligned}
\tau_{0,h} &= -1.443\text{E-}3h^7 - 7.826\text{E-}4h^{10} - 1.139\text{E-}4h^{13} - 7.880\text{E-}6h^{16} + O(h^{19}) \\
\tau_{1,h} &= -4.810\text{E-}4h^6 - 6.080\text{E-}4h^9 - 1.093\text{E-}4h^{12} - 8.514\text{E-}6h^{15} + O(h^{18}) \\
\tau_{2,h} &= -4.642\text{E-}4h^8 - 1.276\text{E-}4h^{11} - 1.310\text{E-}5h^{14} - 6.900\text{E-}7h^{17} + O(h^{20}) \\
\tau_{3,h} &= -1.803\text{E-}4h^7 - 7.020\text{E-}5h^{10} - 7.700\text{E-}6h^{13} - 4.008\text{E-}7h^{16} + O(h^{19}) \\
\tau_{4,h} &= -6.010\text{E-}5h^6 - 5.610\text{E-}5h^9 - 8.420\text{E-}6h^{12} - 5.900\text{E-}7h^{15} + O(h^{18}) \\
\tau_{5,h} &= -6.400\text{E-}5h^8 - 2.220\text{E-}5h^{11} - 2.800\text{E-}6h^{14} - 1.800\text{E-}7h^{17} + O(h^{20}).
\end{aligned}$$

Figure 2 shows function $g(h) := -\tau_{0,h}/(-1.443E - 3\ h^7)$ when $N = 6$. We note that when h approaches 0, function $g(h)$ has a constant-like behavior, as predicted by Theorem 2.

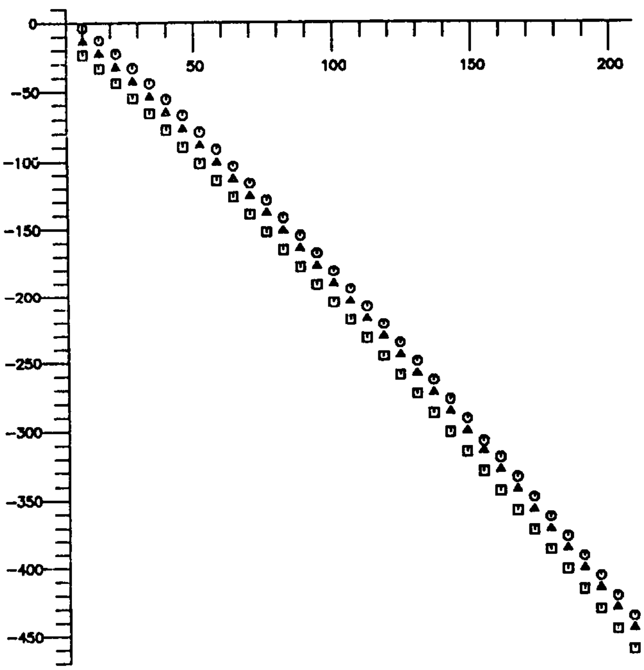

Figure 1. Problem (39)–(40) has been solved by the Tau Method with $H_N(x) = \left(\sum_{i=0}^{5} \tau_{i,N} x^i\right) T_N(x)$; $N = 10(6)208$. The figure shows $\ln \Upsilon(N)$ [$\triangleleft$], $\ln |\tau_{0,N}|$ [$\circ$] and $\ln \|e_N\|_{[-1,1]}$[$\diamond$].

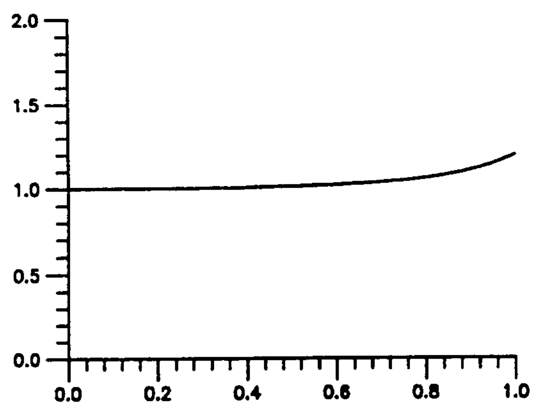

Figure 2. The figure shows the behavior of the Tau parameter $\tau_{j,h}$ in terms of small values of h: $\tau_{j,h} = \theta_j \mathrm{h}^N + O(\mathrm{h}^{N+1})$. For Examples (43)–(44) we have plotted $\tau_{0,h}/(-2.443E - 3h^7)$ in terms of h for $h \leq 1$. Notice that the function plotted is almost constant for $h \ll 1$.

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APPENDIX A1

PROOF OF LEMMA 2:

1. Let $n \geq \alpha$. By (14) n can be written as $n := (\alpha + 2)K_n + \alpha + \vartheta_n$. Let us assume that (16) is true for all $m < n$ and let us prove that it also holds for n: using (7) an induction we can write

$$Q_n(x) = \frac{1}{a} x^{n-\alpha} + \frac{1}{a} \frac{(n-\alpha)!}{(n-\alpha-2)!} Q_{n-\alpha-2}(x)$$

$$= \frac{1}{a}x^{n-\alpha} + \frac{1}{a}\frac{(n-\alpha)!}{(n-\alpha-2)!}$$
$$\left\{\frac{1}{a}x^{n-2\alpha-2} + \sum_{k=1}^{\frac{n-2\alpha-2-\theta_n}{\alpha+2}} \frac{1}{a^{k+1}}\prod_{j=1}^{k}\frac{[(n-j(\alpha+2)-\alpha]!}{[(n-(j+1)(\alpha+2)]!}x^{n-(k+1)(\alpha+2)-\alpha}\right\}$$

$$= \frac{1}{a}x^{n-\alpha} + \frac{1}{a^2}\frac{(n-\alpha)!}{(n-\alpha-2)!}x^{n-2\alpha-2} +$$
$$\left\{\sum_{k=1}^{\frac{n-2\alpha-2-\theta_n}{\alpha+2}} \frac{1}{a^{k+2}}\frac{(n-\alpha)!}{(n-\alpha-2)!}\prod_{j=2}^{k+1}\frac{[n-(j-1)(\alpha+2)-\alpha]!}{[n-j(\alpha+2)]!}x^{n-(k+1)(\alpha+2)-\alpha}\right\}$$

$$= \frac{1}{a}x^{n-\alpha} + \frac{1}{a^2}\frac{(n-\alpha)!}{(n-\alpha-2)!}x^{n-2\alpha-2} +$$
$$\left\{\sum_{k=1}^{\frac{n-2\alpha-2-\theta_n}{\alpha+2}} \frac{1}{a^{k+2}}\prod_{j=1}^{k+1}\frac{[n-(j-1)(\alpha+2)-\alpha]!}{[n-j(\alpha+2)]!}x^{n-(k+1)(\alpha+2)-\alpha}\right\}$$

$$= \frac{1}{a}x^{n-\alpha} + \frac{1}{a^2}\frac{(n-\alpha)!}{(n-\alpha-2)!}x^{n-2\alpha-2} +$$
$$\left\{\sum_{k=2}^{\frac{n-\alpha-\theta_n}{\alpha+2}} \frac{1}{a^{k+1}}\prod_{j=1}^{k}\frac{[n-(j-1)(\alpha+2)-\alpha]!}{[n-j(\alpha+2)]!}x^{n-k(\alpha+2)-\alpha}\right\}$$

$$= \frac{1}{a}x^{n-\alpha} + \sum_{k=1}^{\frac{n-\alpha-\theta_n}{\alpha+2}} \frac{1}{a^{k+1}}\left\{\prod_{j=1}^{k}\frac{[n-(j-1)(\alpha+2)-\alpha]!}{[n-j(\alpha+2)]!}\right\}x^{n-k(\alpha+2)-\alpha}.$$

2. Follows immediately from 1.
3. Can be obtained using similar arguments. ∎

APPENDIX A2

PROOF OF LEMMA 3:

1. Follows from the distributive law of addition.
2. We have

$$\sum_{i=0}^{N} c_{i,h}^{N} x^i := V_N^h(x) = \sum_{j=0}^{N} c_j^N \left(\frac{x-s_0}{\mathbf{h}}\right)^j = \sum_{j=0}^{N}\frac{c_j^N}{\mathbf{h}^j}(x-s_0)^j$$
$$= \sum_{j=0}^{N}\frac{c_j^N}{\mathbf{h}^j}\left[\sum_{i=0}^{j}\binom{j}{i}x^i(-s_0)^{j-i}\right] = \sum_{j=0}^{N}\left[\sum_{i=j}^{N}\frac{c_i^N}{\mathbf{h}^i}\binom{i}{j}(-s_0)^{i-j}\right]x^j.$$

Equating coefficients on both sides we find that

$$c_{j,h}^N = \sum_{i=j}^{N}\frac{c_i^N}{\mathbf{h}^i}\binom{i}{j}(-s_0)^{i-j}; \qquad j = 0,1,\ldots,N, \tag{45}$$

and expanding s_0^{i-j}:

$$s_0^{i-j} = (x_0+\mathbf{h})^{(i-j)} = \sum_{k=0}^{i-j}\binom{i-j}{k}x_0^{i-j-k}\mathbf{h}^k = \sum_{k=0}^{i-j} b_{ik}\,\mathbf{h}^k.$$

Inserting into (45) we get

$$c_{j,h}^N = \sum_{i=j}^{N}\frac{c_i^N}{\mathbf{h}^i}\binom{i}{j}(-1)^{i-j}s_0^{i-j} = \sum_{i=j}^{N}a_i\sum_{k=0}^{i-j}b_{ik}\,\mathbf{h}^k = \sum_{i=0}^{N-j}\left[\sum_{k=i+j}^{N}a_k\,b_{ki}\right]\mathbf{h}^i$$

which gives 2.

3. Substituting (34) in (33)

$$
\begin{aligned}
c_{j,h}^{N} &= \sum_{i=0}^{N-j} \left[\sum_{k=i+j}^{N} \frac{c_k^N}{\mathbf{h}^k} \binom{k}{j} (-1)^{k-j} \binom{k-j}{i} x_0^{k-j-i} \right] \mathbf{h}^i \\
&= \sum_{i=0}^{N-j} \left\{ (-1)^{N-j} \binom{N-j}{i} x_0^{N-j-i} \frac{c_k^N}{\mathbf{h}^N} \binom{N}{j} + O\left(\frac{1}{\mathbf{h}^{N-1}}\right) \right\} \mathbf{h}^i \\
&= \left[(-1)^{N-j} c_N^N \binom{N}{j} x_0^{N-j} \right] \frac{1}{\mathbf{h}^N} + O\left(\frac{1}{\mathbf{h}^{N-1}}\right).
\end{aligned}
$$

∎